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# Correlation functions of the BC Calogero–Sutherland model

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## Abstract

The BC-type Calogero–Sutherland model (CSM) is an integrable extension of the ordinary A-type CSM that possesses a reflection symmetry point. The BC-CSM is related to the chiral classes of random matrix ensembles (RMEs) in exactly the same way as the A-CSM is related to the Dyson classes. We first develop the fermionic replica  $\sigma$ -model formalism suitable to treat all chiral RMEs. By exploiting ‘generalized colour–flavour transformation’ we then extend the method to find the exact asymptotics of the BC-CSM density profile. Consistency of our result with the  $c = 1$  Gaussian conformal field theory description is verified. The emerging Friedel oscillations structure and sum rules are discussed in details. We also compute the distribution of the particle nearest to the reflection point.

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*To the memory of Professor Sung-Kil Yang*

## 1. Introduction

It is a well-known fact that there is an intimate relationship between the one-dimensional quantum problem with the inverse-square interaction potential, i.e. Calogero–Sutherland model (CSM) [1, 2], and Dyson random matrix ensembles (RMEs) [3]. On the most elementary level the correspondence goes as follows: for the three particular values of the coupling constant ( $\lambda = 1/2, 1, 2$ ) square of the many-body ground-state wavefunction of the CSM coincides with the joint probability distribution (JPD) of RMEs with Dyson index  $\beta = 2\lambda$ . Consequently, knowledge of the RME correlation functions may be immediately translated to information about the CSM. Historically this correspondence proved to be very fruitful for advancing the understanding of the CSM.

It was later realized that Dyson's classification of the RMEs was not exhaustive. Studies of the two-sublattice model, mesoscopic transport and QCD Dirac spectra initiated the introduction of their 'chiral' counterparts [4–8]. Subsequently, based on Cartan's classification of Riemannian symmetric spaces, Altland and Zirnbauer [9–11] have further added six 'superconducting' chiral symmetry classes. These chiral classes are characterized by the special role played by the zero energy. Namely, there is a certain number of eigenvalues that *must* have zero energy, while all other eigenvalues occur in symmetric pairs (mirror images) around zero. As a result, the mean density of states (DoS) exhibits either a hollow or a bump around the zero energy followed by decaying oscillations at larger distances. Such a structure on the level of the mean DoS does not show up in the Dyson classes and, as we shall explain below, may be called Friedel oscillations.

The question is whether one can find an appropriate generalization of the CSM whose ground-state wavefunction possesses the same reflection symmetry property as the JPD of chiral ensembles. The answer is known to be affirmative. Indeed, one can write down an integrable one-dimensional model with inverse-square interaction, reflection symmetry and special single-particle potential centred at zero having the required ground state. It is known as the BC-type CSM in the literature [12]. Due to the presence of the mirror boundary and localized single-particle potential (an impurity), the model lacks translational invariance and the resulting ground-state density is not uniform. In particular, the density profile develops the Friedel oscillations far enough from the impurity. Accordingly one has a unique example of integrable strongly interacting models that exhibit Friedel phenomena. That gives one a possibility to gain the exact information on the amplitude decay rate, spectral characteristics and phase shifts of the Friedel oscillations in the interacting system. Despite the proved integrability, explicit form of the correlation functions of the BC-CSM with generic coupling constant were not established, except for some partial results [11, 13–15].

The purpose of this paper is to fill this gap for rational values of the coupling constant. To this end we employ the recently developed approach based on the replica trick [16–18]. It was previously tested on the pair correlation function of the ordinary A-CSM [19], where it perfectly agrees with the exact results of Haldane [20] and Ha [21] for any rational coupling constant. The idea is to explore the relation with the RMEs, where the replica trick was found to be accurate in the asymptotic regime. We thus develop first the fermionic replica approach to the chiral symmetry classes of RMEs. Not surprisingly, we are able to reproduce the asymptotic behaviour of the known DoS profiles for chiral RMEs. We then extend the treatment away from the RMEs values of the coupling constant and obtain closed analytic results for any rational two-body coupling constant  $\lambda$  and any impurity phase shift.

We found that for rational values of  $\lambda = p/q$  ( $p, q$  coprime), the spectrum of the Friedel oscillations contains exactly  $p$  harmonics, corresponding to  $2k_F, 4k_F, \dots, 2pk_F$  density oscillations. The  $l$ th harmonic ( $l = 1, \dots, p$ ) decays algebraically as  $\theta^{-l^2/\lambda}$ , where  $\theta$  is the distance from the impurity. The amplitudes of the harmonics depend on the number,  $l$ , and the coupling constant,  $\lambda$ , but are not sensitive to the strength and details of the impurity expressed through a phase shift,  $\nu$ . Moreover, the harmonics amplitudes are closely related to those of the two-point correlation function of the homogenous A-CSM. We provide a conformal field theory account for this fact. The impurity phase shift,  $\nu$ , affects only phases of the harmonics in the asymptotic regime. We explicitly show that the relation between the total charge expelled by the impurity and the asymptotic phase shifts, known as the Friedel sum rule, holds for the interacting system. We also compute the distribution of the locus of the particle nearest to the mirror boundary point for the BC-CSM at generic values of the coupling constants.

The paper is organized as follows: in section 2 we briefly introduce chiral ensembles and develop the appropriate fermionic replica  $\sigma$ -models. Section 3 is devoted to the introduction and replica treatment of the BC-CSM. In section 4 we perform the analytical continuation and the replica limit and extract the density profile. In section 5 we compare this result with the effective conformal field theory description. In section 6 we compute the nearest particle distribution. A summary and discussions are provided in section 7. Technicalities of the calculations are relegated to the two appendices.

## 2. Chiral circular ensembles

A *circular* RME is defined as an ensemble of unitary matrices  $U$  representing a Riemannian symmetric space  $\mathcal{D}$ , stochastically distributed according to the Haar measure  $dU$  of  $\mathcal{D}$ . Expressing a symmetric space as a coset  $\mathcal{D} = G/H$  with a compact Lie group  $G$  and  $H \subset G$ , the Cartan mapping  $G \rightarrow G/H, g \mapsto U(g)$  is as shown in the table below:

Class	$G$	$H$	$U(g)$
A(CUE)	$U(N)$	1	$g$
AI(COE)	$U(N)$	$O(N)$	$g^T g$
AII(CSE)	$U(2N)$	$Sp(2N)$	$g^D g$
AIII(chCUE)	$U(N + N')$	$U(N) \times U(N')$	$I g^\dagger I g$
BDI(chCOE)	$SO(N + N')$	$SO(N) \times SO(N')$	$I g^T I g$
CII(chCSE)	$Sp(2(N + N'))$	$Sp(2N) \times Sp(2N')$	$I g^D I g$
D, B	$SO(2N), SO(2N + 1)$	1	$g$
C	$Sp(2N)$	1	$g$
CI	$Sp(4N)$	$U(2N)$	$I g^\dagger I g$
DIII <sub>e,o</sub>	$SO(4N), SO(4N + 2)$	$U(2N), U(2N + 1)$	$g^D g$

Here

$$J = \begin{bmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{bmatrix} \quad I = \begin{bmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_{N'} \end{bmatrix} \tag{2}$$

( $N \rightarrow 2N, N' \rightarrow 2N'$  for CII and  $N, N' \rightarrow 2N$  for CI), and  $g^D = J g^T J^{-1}$  denotes the quaternion dual of the matrix  $g$ . Out of these twelve classes, the last nonclassical nine possess chirality [23], i.e. nonzero eigenphases appear in complex conjugate pairs. The JPD of these nonzero eigenphases

$$U = V \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}, 1, \dots, 1) V^\dagger \tag{3}$$

for the circular ensemble is given by [22] ( $0 \leq \theta \leq \pi$ )

$$P(\theta_1, \dots, \theta_N) d\theta_1 \dots d\theta_N = \prod_{i=1}^N \left( d\theta_i \sin^{c_1} \frac{\theta_i}{2} \cos^{c_2} \frac{\theta_i}{2} \right) |\Delta_N(\cos \theta)|^\beta \tag{4}$$

$$= \prod_{i=1}^N (dy_i y_i^{(c_1-1)/2} (1 - y_i)^{(c_2-1)/2}) |\Delta_N(y)|^\beta \quad y_i \equiv \sin^2(\theta_i/2) \tag{5}$$

where  $\Delta_N$  is the Vandermonde determinant of the rank  $N$  and the constants  $c_1$ ,  $c_2$  and  $\beta$  for all nine chiral ensembles are given in the table below.

Class	$c_1$	$c_2$	$\beta$	$\nu$	$\mathcal{D}$ (circular RME)	$\mathcal{M}$ (F-replica NL $\sigma$ M)
AIII	$2(N' - N) + 1$	1	2	$N' - N$	$U(N + N') / (U(N) \times U(N'))$	$U(n)$
BDI	$N' - N$	0	1	$N' - N$	$SO(N + N') / (SO(N) \times SO(N'))$	$U(2n)/Sp(2n)$
CII	$4(N' - N) + 3$	3	4	$N' - N$	$Sp(2(N + N')) / (Sp(2N) \times Sp(2N'))$	$U(2n)/O(2n)$
D	0	0	2	$-1/2$	$SO(2N)$	$SO(2n)/U(n)$
B	2	0	2	$1/2$	$SO(2N + 1)$	$SO(2n)/U(n)$
C	2	2	2	$1/2$	$Sp(2N)$	$Sp(2n)/U(n)$
CI	1	1	1	1	$Sp(2N)/U(N)$	$Sp(2n)$
DIII <sub>e</sub>	1	1	4	$-1/2$	$SO(4N)/U(2N)$	$SO(2n)$
DIII <sub>o</sub>	5	1	4	$1/2$	$SO(4N + 2)/U(2N + 1)$	$SO(2n)$

The JPD equation (5) defined over  $y \in [0, 1]$  may be comprehensively called Jacobi ensemble. The constant  $\nu$ , defined as

$$\nu = \frac{c_1 + 1}{\beta} - 1 \quad (7)$$

may be called ‘topological charge’, when the tangent element of a random matrix  $U$  in the first three cases is interpreted as the QCD Dirac operator in even dimensions [6, 8].

The real characteristic polynomial, or so called the fermionic replicated partition function, for these circular ensembles is defined as

$$Z_{n,N}(\theta) = \int_{\mathcal{D}} dU \det(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}U)^n \equiv \int_G dg \det(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}U(g))^n. \quad (8)$$

After the colour–flavour transformation [23] and the thermodynamic limit where  $N \rightarrow \infty$ ,  $\theta \rightarrow 0$  with their product (denoted by the same  $\theta$  for the sake of simplicity) fixed finite, it takes the form

$$Z_n(\theta) \equiv \lim_{N \rightarrow \infty} Z_{n,N} \left( \frac{\theta}{N} \right) = \theta^{nv} \int_{\mathcal{M}} du (\det u)^v e^{i\frac{\theta}{2} \text{tr}(u+u^\dagger)} \quad (9)$$

where  $\mathcal{M}$  are the ‘dual’ symmetric spaces of the fermionic nonlinear  $\sigma$ -models which are listed in table (6). Hereafter we suppress irrelevant normalization constants that go to unity in the replica limit,  $n \rightarrow 0$ . The derivation of equation (9) from equation (8) for the AIII, BDI and CII classes is summarized in appendix A. In order to derive equation (9) for symmetry classes whose pertinent colour–flavour transformations are not immediately available in the literature, one could also adopt an alternative method of magnifying the origin of the circular ensembles first (i.e. employing Gaussian ensembles), performing Hubbard–Stratonovich transformation and taking the thermodynamic limit. For a Gaussian ensemble treatment of the BDI and CII classes, see [24].

Performing the integration over angular degrees of freedom  $v$  of  $u = v \text{diag}(e^{i\phi_a})v^\dagger$ , one obtains

$$Z_n(\theta) = \theta^{nv} \int_0^{2\pi} \prod_{a=1}^n (d\phi_a e^{iv\phi_a + i\theta \cos \phi_a}) \left| \prod_{a>b}^n \sin \left( \frac{\phi_a - \phi_b}{2} \right) \right|^{4/\beta}. \quad (10)$$

This expression is valid for all nine chiral symmetry classes. Note that equation (10) depends only on  $\beta$  and  $c_1$  but not on  $c_2$ , because we have magnified the vicinity of the origin,  $\theta = 0$ . Accordingly the class C gives the same  $Z_n(\theta)$  as B, and the class CI the same as BDI at  $c_1 = 1$ , reducing nine symmetry classes of chiral RMEs to seven universality classes. The number could be further reduced by introducing three universality classes of Laguerre ensembles, having continuous  $\nu$  and discrete  $\beta = 2, 1, 4$ .

Consequently, we succeeded in expressing the integration over the initial  $N$ -variable JPD, equation (4), through the  $n$ -fold integral, equation (10). We shall discuss its evaluation, analytical continuation and the replica limit after we introduce the BC-CSM. This replica treatment was previously performed for the AIII class in [25].

### 3. BC Calogero–Sutherland model

The generalized Calogero–Sutherland Hamiltonian [12] is defined as

$$H = - \sum_{i=1}^N \frac{\partial^2}{\partial \theta_i^2} + \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}}{\sin^2(\theta \cdot \alpha/2)}. \quad (11)$$

Here  $\Delta$  is a root system of a Lie algebra in  $N$ -dimensional vector space,  $\theta$  is the vector  $(\theta_1, \dots, \theta_N)$  and  $g_{|\alpha|}$  is a coupling constant depending only on the root length. Quantum integrability is ensured by these conditions. The ordinary, translationally invariant model corresponds to the  $A_{N-1}$  root system.

The quantum one-dimensional model of  $N$  interacting particles on a semicircle,  $0 \leq \theta_i \leq \pi$ , where  $i = 1, \dots, N$ , with the Hamiltonian

$$H = - \sum_{i=1}^N \frac{\partial^2}{\partial \theta_i^2} + \frac{\lambda(\lambda-1)}{2} \sum_{i>j}^N \left[ \frac{1}{\sin^2 \frac{\theta_i - \theta_j}{2}} + \frac{1}{\sin^2 \frac{\theta_i + \theta_j}{2}} \right] \\ + \frac{\lambda_1(\lambda_1-1)}{4} \sum_{i=1}^N \frac{1}{\sin^2 \frac{\theta_i}{2}} + \frac{\lambda_2(\lambda_2-1)}{4} \sum_{i=1}^N \frac{1}{\cos^2 \frac{\theta_i}{2}} \quad (12)$$

corresponds to the  $BC_N$  root system, hence called the BC-CSM. As the  $BC_N$  root system contains roots of length 1,  $\sqrt{2}$ , 2, there are three independent coupling constants,  $\lambda, \lambda_1, \lambda_2$ . This family contains CSMs corresponding to  $B_N(\lambda_2 = 0)$ ,  $C_N(\lambda_1 = \lambda_2)$  and  $D_N(\lambda_1 = \lambda_2 = 0)$  root systems as its subfamilies. In addition to the pairwise inverse-square interactions, the particles interact with their own mirror images (occupying the other semicircle,  $\pi \leq \theta \leq 2\pi$ ) and with the two single-particle impurity potentials placed at  $\theta = 0$  and  $\theta = \pi$ . In what follows we shall assume the thermodynamic limit,  $N \rightarrow \infty$ , and shall focus on the vicinity of the impurity at  $\theta = 0$ . The other impurity may be treated in exactly the same manner. The model is known to have the following ground-state energy [12]:

$$H\Psi_0 = E_0\Psi_0 \quad E_0 = \sum_{i=1}^N \left( (N-i)\lambda + \frac{\lambda_1 + \lambda_2}{2} \right)^2 \quad (13)$$

and the ground-state wavefunction

$$\Psi_0(\theta_1, \dots, \theta_N) = \prod_{i=1}^N \left( \sin^{\lambda_1} \frac{\theta_i}{2} \cos^{\lambda_2} \frac{\theta_i}{2} \right) \Delta_N(\cos \theta)^\lambda. \quad (14)$$

The absolute square of the ground-state wavefunction coincides with the JPD of the chiral RMEs (4) [26], through a change of the coupling constants

$$\lambda = \beta/2 \quad \lambda_1 = c_1/2 \quad \lambda_2 = c_2/2. \quad (15)$$

Note that equation (14) is *not* restricted to the special values of  $\beta, c_1, c_2$  listed in table (6). The energy and wavefunctions of the excited states were studied in [27–30].

The particle density is defined as

$$\langle \rho(\theta) \rangle \equiv \left\langle \sum_{j=1}^N \delta(\theta - \theta_j) \right\rangle = \sqrt{y(1-y)} \left\langle \sum_{j=1}^N \delta(y - y_j) \right\rangle \quad (16)$$

where  $y \equiv \sin^2(\theta/2)$ . The angular brackets denote ground-state expectation values, or equivalently averaging over the normalized JPD, equations (4) and (5), for the first and second equalities correspondingly. One may then employ the replica trick to write

$$\sum_{j=1}^N \delta(y - y_j) = \lim_{n \rightarrow 0} \frac{1}{n\pi} \operatorname{Im} \frac{d}{dy} \prod_{j=1}^N (y - y_j - i\epsilon)^n. \quad (17)$$

As a result, one obtains

$$\langle \rho(\theta) \rangle = \sqrt{y(1-y)} \lim_{n \rightarrow 0} \frac{1}{n\pi} \operatorname{Im} \frac{d}{dy} Z_{n,N}(y - i\epsilon) \Big|_{y=\sin^2(\theta/2)} \quad (18)$$

where the ‘replicated partition function’ is defined as

$$Z_{n,N}(y) = \int_0^1 \prod_{i=1}^N (dy_i y_i^{\lambda_1-1/2} (1-y_i)^{\lambda_2-1/2} (y-y_i)^n) (\Delta_N(y)^2)^\lambda. \quad (19)$$

Baker and Forrester [31] have noted the integral equality due to Kaneko [32] and Yan [33], which we suggestively call the ‘generalized colour–flavour transformation’. With its help, one may express the partition function in the following way:

$$Z_{n,N}(y) = \int_{\mathcal{C}} \prod_{a=1}^n \left( dx_a x_a^{\frac{\lambda_1+\lambda_2+1}{\lambda}-2} (1-x_a)^{-\frac{\lambda_2+n-1/2}{\lambda}} \left[ \frac{x_a(1-yx_a)}{1-x_a} \right]^N \right) (\Delta_n(x)^2)^{1/\lambda}. \quad (20)$$

The integration contour  $\mathcal{C}$  encircles the cut between  $x_a = 0$  and  $x_a = 1$ . The general form of the integral identity is given in appendix B. So far no approximation has been made. Now we pass to the thermodynamic limit,  $N \rightarrow \infty$ , and magnify the vicinity of the  $\theta = 0$  impurity. To this end we rescale the variable as  $\theta \rightarrow \theta/N$  and correspondingly  $y \simeq \theta^2/(4N^2)$ . By redefining the integration variables as  $x_a = 1 - 2iN\theta^{-1} e^{i\phi_a}$  and taking the thermodynamic limit,  $N \rightarrow \infty$ , one finds (see appendix B for details)

$$\begin{aligned} Z_n(\theta) &\equiv \lim_{N \rightarrow \infty} Z_{n,N} \left( \frac{\theta}{N} \right) \\ &= \theta^{n(\frac{\lambda_1}{\lambda} + \frac{1}{2\lambda} - 1)} \int_0^{2\pi} \prod_{a=1}^n \left( d\phi_a e^{i(\frac{\lambda_1}{\lambda} + \frac{1}{2\lambda} - 1)\phi_a + i\theta \cos \phi_a} \right) \left[ \prod_{a>b}^n \sin^2 \left( \frac{\phi_a - \phi_b}{2} \right) \right]^{1/\lambda}. \end{aligned} \quad (21)$$

One may introduce notation

$$\nu = \frac{\lambda_1}{\lambda} + \frac{1}{2\lambda} - 1 \quad (22)$$

to note the exact coincidence with the  $\sigma$ -model representation of the chiral RMEs, equation (10), provided that the coupling constants are related via equation (15). The important difference is that the BC-CSM representation in the form of equation (21) is *not* restricted to the RMEs values  $\lambda = 1/2, 1, 2$  and special values of the topological charge,  $\nu$ .

#### 4. Analytic continuation and replica limit

Consider the  $n$ -fold integral

$$Z_n(\theta) = \theta^{nv} \int_0^{2\pi} \prod_{a=1}^n (d\phi_a e^{iv\phi_a + i\theta \cos \phi_a}) \left[ \prod_{a>b}^n \sin^2 \left( \frac{\phi_a - \phi_b}{2} \right) \right]^{1/\lambda}. \quad (23)$$

One may stretch the integration contour from the unit circle in the complex plane of  $z_a = e^{i\phi_a}$  into two lines parallel to the imaginary axis with  $\text{Re } z_a = \pm 1$ . The original integral, equation (23), splits into the sum of  $n$  terms with  $l$  integrals having  $\text{Re } z_a = -1$  and remaining  $n - l$  ones  $\text{Re } z_a = 1$ ; here  $l = 1, \dots, n$ . The further progress is made possible in the asymptotic limit,  $\theta \gg 1$ . In this case, the integrals are dominated by the vicinities of the saddle points  $z_a = \pm 1$  and thus may be evaluated employing the Selberg integral. This strategy was described in details in [16–19]. Proceeding this way, one finds for the replicated partition function

$$Z_n(\theta) = \theta^{nv} e^{in\theta} \sum_{l=0}^n F_n^l(\lambda) e^{i\pi vl - 2il\theta} 2^{-\frac{2l^2}{\lambda}} (i\theta)^{-\frac{n-l}{2}(1+\frac{n-l}{\lambda})} (-i\theta)^{-\frac{l}{2}(1+\frac{l-1}{\lambda})} \quad (24)$$

where we have omitted a normalization constant that goes to unity in the replica limit,  $n \rightarrow 0$  and, following [16–19], introduced the notation

$$F_n^l(\lambda) \equiv \binom{n}{l} \prod_{a=1}^l \frac{\Gamma(1 + a/\lambda)}{\Gamma(1 + (n - a + 1)/\lambda)}. \quad (25)$$

Employing the observation that  $F_n^l(\lambda) \equiv 0$  for  $l > n$ , one may extend summation over  $l$  in equation (24) to infinity and then perform the analytic continuation  $n \rightarrow 0$ . As a result, one finds via  $\langle \rho(\theta) \rangle = \lim_{n \rightarrow 0} (\pi n)^{-1} \text{Im } \partial_\theta Z_n(\theta)$  (cf equation (18)),

$$\langle \rho(\theta) \rangle = \frac{1}{\pi} \left[ 1 + 2 \sum_{l=1}^{\infty} \frac{d_l(\lambda)}{(2\theta)^{l^2/\lambda}} \cos \left( 2l\theta - l\pi \left( \nu + \frac{1}{2} - \frac{1}{2\lambda} \right) \right) \right] \quad (26)$$

where

$$d_l(\lambda) \equiv \frac{(-1)^l}{2^{l^2/\lambda}} \prod_{a=1}^l \frac{\Gamma(1 + a/\lambda)}{\Gamma(1 - (a - 1)/\lambda)}. \quad (27)$$

Equation (26) for the asymptotic of the ground-state density of the BC-CSM is the central result of this paper. For the integer values of the coupling constant,  $\lambda$ , it may be extracted from the expressions derived by Baker and Forrester [31], see also [15].

We remark that each term in equation (26) is the one with the lowest power in  $\theta^{-1}$  among all terms carrying the same frequency  $2l$ . This corresponds to truncating all contributions from the descendent fields in the conformal description, see the next section. One could compute these secondary terms, subleading in  $\theta$ , by performing a perturbative expansion around each saddle point with the help of the loop equations [18].

#### 5. Conformal field theory description

By comparing the one-particle correlation function of the BC-CSM (26) with the equal-time *two-particle* correlation function of the ordinary A-CSM [19, 21]

$$\langle \rho(\theta) \rho(0) \rangle = \frac{1}{(2\pi)^2} \left[ 1 - \frac{1}{2\lambda\theta^2} + 2 \sum_{l=1}^{\infty} \frac{d_l(\lambda)^2}{\theta^{2l^2/\lambda}} \cos(2l\theta) \right] \quad (28)$$



one notes that it is expressed through the very same coefficients  $d_l(\lambda)$ , equation (27). It is thus clear that the harmonics amplitudes,  $d_l(\lambda)$ , are properties of the homogenous interacting system and *not* of the localized impurity. As explained below, this result could be anticipated from the effective conformal field theory description. The latter is capable of predicting the low-energy properties of the system apart from numerical values of the coefficients.

Based on the finite-size scaling analysis, Kawakami and Yang [34] identified the low-energy effective theory of the CSM in the thermodynamic limit to be the  $c = 1$  Gaussian conformal field theory at radius  $R = \sqrt{\lambda}/2$ , either nonchiral (A-CSM)

$$\mathcal{L} = \frac{1}{2\pi} \partial_z \Phi \partial_{\bar{z}} \Phi \quad \Phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z}) \quad \Phi \equiv \Phi + 2\pi R \quad (29)$$

or chiral (BC-CSM) [14, 26]. Namely, they have found the density operator should have zero winding number. It does not have a definite conformal dimension, and therefore is expanded in terms of primary and secondary operators whose left- and right-moving vertex momenta are equal. Here we shall consider only contributions from the U(1) current and the primary fields (vertex operators with charges allowed by the compactification (29)),

$$\rho(z, \bar{z}) = \rho_0 \left[ b(\partial_z \phi(z) + \partial_{\bar{z}} \bar{\phi}(\bar{z})) + \sum_{l=-\infty}^{\infty} d_l e^{il(z+\bar{z})} e^{i\phi(z)/R} e^{i\bar{\phi}(\bar{z})/R} \right] \quad (30)$$

where the expansion coefficients,  $b$  and  $d_l (= d_{-l})$ , are not determined from the conformal field theory. Neither are the oscillation factors  $e^{il(z+\bar{z})}$ , which are set by hand to describe the transport of  $l$  pseudo-particles [21] from the left Fermi point ( $-k_F = -1$  by normalization) to the right Fermi point ( $k_F = 1$ ) [34]. We have factored out  $\rho_0$  such that the constant term carries  $d_0 = 1$ . The propagator and the vertex correlator are given by

$$\langle \phi(z) \phi(z') \rangle = -\frac{1}{4} \log(z - z') \quad \langle e^{i\phi(z)/R} e^{i\phi(z')/R} \rangle = \frac{\delta_{l,-l'}}{(z - z')^{l^2/(4R^2)}}. \quad (31)$$

The coefficient in the second equation is a matter of convention and reflects a particular choice of the ultraviolet regularization. Another choice of the regularization would change coefficients  $d_l$ , but not the final result. Employing equations (30) and (31), one obtains for the equal-time two-particle correlation function of the A-CSM (we denote  $z = \theta + i\tau$ )

$$\langle \rho(\theta + i0) \rho(\theta' + i0) \rangle = \rho_0^2 \left[ -\frac{b^2}{2(\theta - \theta')^2} + \sum_{l=-\infty}^{\infty} d_l^2 \frac{e^{2il(\theta - \theta')}}{(\theta - \theta')^{l^2/(2R^2)}} \right]. \quad (32)$$

Comparing this expression with equation (28), one finds that the  $b = 1/\sqrt{\lambda}$  [21], while  $d_l = d_l(\lambda)$ , cf equation (27). Let us consider now the BC-CSM and concentrate on the case without the phase shift for simplicity. The Dirichlet boundary condition at  $\text{Re } z = 0$  is translated into the open boundary bosonization rule [35]

$$\phi(z) = -\bar{\phi}(z) \quad \text{for } \text{Im } z < 0. \quad (33)$$

As the right mover is identified as  $(-1)$  times the left mover at the mirror-imaged point, there exists a nonzero matrix element between the left- and right-moving vertex operators

$$\langle e^{i\phi(z)/R} e^{i\bar{\phi}(\bar{z}')/R} \rangle = \frac{\delta_{l,l'}}{(z + \bar{z}')^{l^2/(4R^2)}}. \quad (34)$$

As a result, the mean density of the BC-CSM becomes nontrivial,

$$\langle \rho(\theta + i0) \rangle = \rho_0 \sum_{l=-\infty}^{\infty} d_l \frac{e^{2il\theta}}{(2\theta)^{l^2/(4R^2)}}. \quad (35)$$

It is in exact agreement with equation (26) if one disregards the phase shift. The phase shift may also be included in the conformal description by shifting the identification of the right and left movers, equation (33), by a constant factor:  $-\pi(\nu + 1/2 - 1/(2\lambda))$ .

We found that knowledge of the asymptotic behaviour of the correlator of the homogenous A-CSM, equation (28), supplemented by the conformal field theory description is, in principle, sufficient to predict the BC-CSM correlation function, equation (26). This agreement indicates the fact that the ultraviolet property of the field that is responsible for the normalization of vertex operators are not affected by the presence or absence of the boundary. Well anticipated as it is, we nevertheless consider this fact to be worth verifying, as done in this paper. This fact can be put on a further test by computing the asymptotics of, e.g., two-particle correlation function for the BC-CSM, though it is technically more challenging. We could as well reverse the logic and conjecture the asymptotically expanded form of any  $p$ -point correlation function of density operators for the A-CSM ( $p \geq 3$ ) or for the BC-CSM ( $p \geq 2$ ), by using equations (30), (31) and (27).

## 6. Nearest particle distribution

Another quantity of interest in the theory of interacting electrons is the probability  $E[s', s]$  of finding no particle within an interval  $[s', s]$  or the particle spacing distribution  $p(s)$  that is a derivative of the former. In the context of spin chains a similar quantity was recently discussed in [36]. For nonchiral as well as chiral RMEs, Tracy and Widom [37] has developed Mehta's computation [3] of  $E[s', s]$  as a Fredholm determinant into a systematic and powerful method. As their method determines  $E[s', s]$  as a solution ( $\tau$  function) to a transcendental equation of Painlevé type relies upon the orthogonal polynomials, its validity is necessarily limited to  $\lambda = 1/2, 1, 2$ . On the other hand,  $E(s) \equiv E[0, s]$  for the chiral RMEs has been computed by an alternative and far simpler 'shifting' method [5, 7, 13, 38, 39] as explained below. We show that with a help of the generalized colour–flavour transformation, this method is applicable also to the BC-CSM at generic values of the coupling constants. Consider for a moment  $\lambda_1 = n + 1/2$ , where  $n = 0, 1, 2, \dots$  is an integer and  $\lambda_2 = 1/2$ . As we are interested in the universal behaviour in the vicinity of the reflection point  $\theta = 0$ , the restriction on  $\lambda_2$  is irrelevant. The probability of having no particle within an interval  $0 \leq \theta \leq s$ , or  $0 \leq y \leq Y$  with  $Y = \sin^2(s/2)$ , is defined as

$$E_N(s) = \text{const} \int_Y^1 \prod_{i=1}^N (dy_i y_i^n) (\Delta_N(y)^2)^\lambda. \quad (36)$$

The constant should be chosen to ensure  $E_N(0) = 1$ . By shifting and rescaling the integration variable as  $y \rightarrow (1 - Y)y + Y$ , one obtains

$$E_N(s) = \text{const} (1 - Y)^{(1+n)N + \lambda N(N-1)} \int_0^1 \prod_{i=1}^N \left[ dy_i \left( y_i + \frac{Y}{1 - Y} \right)^n \right] (\Delta_N(y)^2)^\lambda. \quad (37)$$

Now we apply the generalized colour–flavour transformation (20) to reexpress  $E_N(s)$  as an  $n$ -fold integral,

$$E_N(s) = \text{const} (1 - Y)^{(1+n)N + \lambda N(N-1)} \times \int_{\mathcal{C}} \prod_{a=1}^n \left[ dx_a x_a^{\frac{2}{\lambda} - 2} (1 - x_a)^{-\frac{n}{\lambda}} \left[ \frac{x_a (1 + \frac{Y}{1 - Y} x_a)}{1 - x_a} \right]^N \right] (\Delta_n(x)^2)^{1/\lambda}. \quad (38)$$

We finally rescale  $s \rightarrow s/N$ ;  $Y \rightarrow s^2/(4N^2)$  and take the thermodynamic limit. Following the same procedure that lead from equation (20) to (21), one finds

$$E(s) \equiv \lim_{N \rightarrow \infty} E_N \left( \frac{s}{N} \right) \\ = \text{const} e^{-\frac{\lambda}{4}s^2} s^{n(1-\frac{1}{\lambda})} \int_0^{2\pi} \prod_{a=1}^n (d\phi_a e^{i(\frac{1}{\lambda}-1)\phi_a + s \cos \phi_a}) \left[ \prod_{a>b}^n \sin^2 \left( \frac{\phi_a - \phi_b}{2} \right) \right]^{1/\lambda}. \quad (39)$$

This exact result has previously been derived from Laguerre (chiral Gaussian) ensembles at arbitrary  $\lambda$  and at  $\lambda_1 = n + 1/2$  [13].

To compute the asymptotics of  $E(s)$  for  $s \gg 1$ , one can relax the restriction on  $\lambda_1$ , by first evaluating the  $n$ -fold integral (39) by the saddle point method and then performing the analytical continuation  $n \rightarrow \lambda_1 - 1/2$ . In the large- $s$  limit we pick only the contribution of the replica-symmetric saddle point  $\phi_a = 0$ , since the contribution of all other saddle points is exponentially smaller (the difference with the previous computation is that  $s$  enters equation (39) without imaginary unit). This way Forrester [13] has derived the asymptotic ( $s \gg 1$ ) result

$$E(s) = \text{const} s^{\frac{n}{2} - \frac{n(n+1)}{2\lambda}} e^{-\frac{\lambda}{4}s^2} e^{ns} \\ = \text{const} s^{-\nu(\lambda_1 - \frac{1}{2})} e^{-\frac{\lambda}{4}s^2 + (\lambda_1 - \frac{1}{2})s} \quad (40)$$

where  $\nu$  is the topological charge defined by equation (22). The Gaussian factor  $\exp(-\lambda s^2/4)$  could be anticipated from the mean-field treatment of the classical logarithmic gas [36]. The other factors in the asymptotic expression equation (40) could not be found in any simpler way, to the best of our knowledge.

The distribution  $p(s)$  of the locus  $s$  of the particle nearest to the reflection point is given by  $p(s) = -\partial_s E(s)$ . In the other limiting case,  $s \ll 1$ ,  $p(s)$  is determined by the interaction of the particle closest to the reflection point to its own mirror image. Inspecting equation (5) one immediately finds  $p(s) \propto s^{2\lambda_1}$ .

## 7. Discussions

Let us now take a closer look at our main result, equation (26). The constant term on its rhs,  $\rho_0 = 1/\pi$ , represents the uniform density of particles ( $N$  particles within  $[0, \pi]$ ) far away from the impurity. It may be traced back to the replica symmetric contribution (all  $n$  integrals are taken at  $z_a = -1$  saddle point) to the partition function. The decaying (as  $\theta \rightarrow \infty$ ) oscillatory terms on the rhs of equation (26) are the Friedel oscillations of the particle density induced by the mirror boundary and the impurity potential. These terms may be identified as the replica symmetry broken contributions to the partition functions ( $l$  integrals are taken at the ‘wrong’ saddle point,  $z_a = 1$ ). In general, there is an infinite number of harmonics (unlike a single ‘ $2k_F$ ’ harmonic in the noninteracting system!) in the oscillation spectrum. Since  $d_l \propto \exp(\lambda^{-1} l^2 \ln l)$  for  $l \gg 1$ , the sum over harmonics on the rhs of equation (26) is, in general, not convergent. It is not clear to us at the moment, whether there is a consistent resummation scheme. There is, however, an important class of the parameters, where equation (26) is mathematically rigorous.

For any rational coupling constant  $\lambda = p/q$  the coefficient  $d_l(p/q) \equiv 0$  for  $l > p$  and therefore the sum terminates after exactly  $p$  oscillatory components. One finds that, in addition to the usual  $2k_F$  Friedel oscillation, the system possess  $4k_F, \dots, 2pk_F$  oscillatory components of the density (in the unit  $k_F = 1$  accepted here). This fact might be expected from the form

of the density–density response function of the homogenous A-CSM [20, 21]. However, the algebraic decay rate of the harmonics could *not* be determined employing linear response of the A-CSM. Indeed, the latter predicts that the  $l$ th harmonic decays as  $\theta^{-2l^2/\lambda+1}$ , while the correct decay rate is  $\theta^{-l^2/\lambda}$ . Note that for noninteracting particles,  $\lambda = 1$  and thus  $l = 1$ , both ways give the correct one-dimensional decay of the  $2k_F$  Friedel oscillations:  $\theta^{-1}$ . For any interacting system,  $\lambda \neq 1$ , the linear response is bound to fail in the asymptotic regime. These observations were already made in the Tomonaga–Luttinger liquid literature [40, 41]. Now we can confirm them having the exactly solvable model system.

In the BC-CSM one finds not only the decay law, but also the relative amplitudes of the harmonics: coefficients  $d_l(\lambda)$ , equation (27). Note that these amplitudes are determined by the interaction strength,  $\lambda$ , only and are independent of the impurity strength,  $\lambda_1$ . This is due to the fact that the mirror boundary condition induces oscillations of the maximal possible amplitude. The additional single-particle potential centred at  $\theta = 0$  and characterized by  $\lambda_1$  changes the phase of the oscillations only. The entire information about the impurity strength,  $\lambda_1$ , is incorporated in the parameter  $\nu$ , equation (22). In the asymptotic regime, the latter affects the phase of the Friedel oscillations only and therefore may be associated with the impurity phase shift. (Unlike the leading order, the amplitudes of subleading perturbative corrections in negative powers of  $\theta$  do depend on the phase shift,  $\nu$ .)

To verify equations (26) and (27) one may compare them with the available exact DoS of the chiral RMEs (see [42] and references therein). Employing Hankel’s asymptotic expansion of the Bessel function, one may check that these asymptotic perfectly agree with equations (26) and (27). One may also note that for D, B and C symmetry classes we have obtained the exact rather than the asymptotic results. This coincidence is due to the Duistermaat–Heckman localization theorem [43, 44]. We see that having unitary symmetry class,  $\beta = 2$ , is not sufficient to satisfy the Duistermaat–Heckman theorem. One should also have the special value of the topological charge,  $\nu = \pm 1/2$ , to secure cancellation of all higher-order perturbative corrections [25].

One may define the total charge attracted (expelled) by the impurity to (from) the region near  $\theta = 0$  as

$$Q \equiv \int_0^\infty d\theta (\rho(\theta) - \rho_0) \quad (41)$$

where  $\rho_0 = 1/\pi$  is the uniform asymptotic density. For the chiral RMEs, where the exact expressions including small  $\theta$  region are available, the result is (cf equation (26))

$$Q = -\frac{1}{2} \left( \nu + \frac{1}{2} - \frac{1}{2\lambda} \right) = \frac{1}{4} - \frac{\lambda_1}{2\lambda}. \quad (42)$$

Note that the attracted charge depends both on the impurity amplitude,  $\lambda_1$ , and the interaction strength,  $\lambda$ , reflecting the fact that the impurity is screened due to the interactions. The pure mirror boundary, without the single-particle potential attracts quarter of a particle irrespective to the interaction strength. Equation (42) is a manifestation of the famous Friedel sum rule: the total expelled charge is equal to the impurity phase shift (divided by  $2\pi$ ); the latter also determines the phase of the density oscillations far from the impurity. We conjecture, in accordance with the earlier works [45], that equation (42) is valid for any values of  $\lambda$  and  $\lambda_1$ .

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### Appendix A. $\sigma$ -model derivation via colour–flavour transformation

We first consider the simplest case AIII with  $\nu = 0$ . The Grassmannian  $U(2N)/(U(N) \times U(N))$  is a complex Kähler manifold, and its unitary matrix representative  $U$  in table (1) is conveniently parametrized by the complex stereographic coordinate  $Z_{ij}$ ,  $i, j = 1, \dots, N$ , as

$$U = I g^\dagger I g = I \gamma I \gamma^{-1} \quad \gamma = \begin{bmatrix} 1 & Z \\ -Z^\dagger & 1 \end{bmatrix} \quad Z \in \mathbb{C}^{N \times N} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (\text{A1})$$

The Kähler potential  $k(Z, \bar{Z}) = \text{tr} \log(1 + ZZ^\dagger)$  leads to the Haar measure

$$dU = \frac{\prod_{i,j=1}^N d^2 Z_{ij}}{\det(1 + ZZ^\dagger)^{2N}}. \quad (\text{A2})$$

The replicated partition function then reads

$$\begin{aligned} Z_{n,N}(\theta) &= \int_{U(2N)/(U(N) \times U(N))} dU \det(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} U)^n \\ &= \int_{\mathbb{C}^{N \times N}} \frac{\prod d^2 Z}{\det(1 + ZZ^\dagger)^{2N}} \det(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} I \gamma I \gamma^{-1})^n \\ &= \int_{\mathbb{C}^{N \times N}} \frac{\prod d^2 Z}{\det(1 + ZZ^\dagger)^{2N+n}} \det(e^{i\frac{\theta}{2}} \gamma - e^{-i\frac{\theta}{2}} I \gamma I)^n. \end{aligned} \quad (\text{A3})$$

We introduce a set of  $(N \times n)$ -component independent Grassmannian numbers  $\psi_i^a, \chi_i^a, \bar{\psi}_i^a, \bar{\chi}_i^a, i = 1, \dots, N, a = 1, \dots, n$ , to exponentiate the determinant ( $I \gamma I = \gamma^\dagger$ ),

$$Z_{n,N}(\theta) = \int d\psi d\bar{\psi} d\chi d\bar{\chi} \int_{\mathbb{C}^{N \times N}} \frac{\prod d^2 Z}{\det(1 + ZZ^\dagger)^{2N+n}} \exp\left([\bar{\psi} \bar{\chi}] (e^{i\frac{\theta}{2}} \gamma - e^{-i\frac{\theta}{2}} \gamma^\dagger) \begin{bmatrix} \psi \\ \chi \end{bmatrix}\right). \quad (\text{A4})$$

Now we employ Zirnbauer's colour–flavour transformation [23]

$$\int_{\mathbb{C}^{N \times N}} \frac{\prod d^2 Z}{\det(1 + ZZ^\dagger)^{2N+n}} \exp(\bar{\psi}_i^a Z_{ij} \chi_j^a - \bar{\chi}_i^a Z_{ij}^\dagger \psi_j^a) = \int_{U(n)} du \exp(\bar{\psi}_i^a u^{ab} \psi_i^b + \bar{\chi}_i^a u^{\dagger ab} \chi_i^b) \quad (\text{A5})$$

to obtain

$$\begin{aligned} Z_{n,N}(\theta) &= \int d\psi d\bar{\psi} d\chi d\bar{\chi} \\ &\quad \times \int_{U(n)} du \exp\left(\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)(\bar{\psi} \psi + \bar{\chi} \chi) + \left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right)(\bar{\psi} u \psi + \bar{\chi} u^\dagger \chi)\right) \\ &= \int_{U(n)} du \det\left(\cos \theta + i \sin \theta \frac{u + u^\dagger}{2}\right)^N. \end{aligned} \quad (\text{A6})$$

We stress that no approximation has been made in the above procedure.

In the thermodynamic limit,  $N \rightarrow \infty$  and  $\theta \rightarrow 0$ , one has  $\cos \theta \simeq 1$  and  $\sin \theta \simeq \theta$ . As a result, the determinant in equation (A6) may be exponentiated:

$$Z_n(\theta) \equiv \lim_{N \rightarrow \infty} Z_{n,N}\left(\frac{\theta}{N}\right) = \int_{U(n)} du e^{i\frac{\theta}{2} \text{tr}(u + u^\dagger)}. \quad (\text{A7})$$

One can repeat the above procedure for the BDI and CII classes. The parametrization (A1) of the real and quaternionic Grassmannian manifolds,  $SO(2N)/(SO(N) \times SO(N))$  and  $Sp(4N)/(Sp(2N) \times Sp(2N))$ , involves  $N \times N$  real and quaternion-real matrices  $Z$ , instead of complex. The colour–flavour transformation trades the integrations over these ‘coloured’ variables  $Z$  with the ones over ‘flavoured’ variables  $u$  that are antisymmetric and symmetric unitary matrices, respectively (straightforward as they are, such types of colour–flavour transformation have yet to be exhibited explicitly in the literature, to the best of our knowledge). Accordingly, the integration domain of the transformed partition function (A6) becomes antisymmetric unitary matrices ( $U(2n)/Sp(2n)$ ) and symmetric unitary matrices ( $U(2n)/O(2n)$ ), with the rest being unaltered. Inclusion of nonzero  $\nu$  is straightforward by considering a rectangular  $Z$ . It merely shifts the the power of the determinant in the measure (A2) by  $\nu$ , and modifies equation (A7) into equation (9) in the thermodynamic limit.

For the B-D and C or DIII and CI classes, one utilizes the colour–flavour transformation between the orthogonal or symplectic group and the associated dual symmetric space parametrized by antisymmetric or symmetric complex matrices, respectively [23, 46, 47].

## Appendix B. Generalized colour–flavour transformation

Kaneko [32] (see also Yan [33]) has derived the following remarkable integral identity:

$$\begin{aligned} Z_{n,N}(t) &= \frac{1}{S_N(\Lambda_1 + n, \Lambda_2, \lambda)} \int_0^1 \prod_{i=1}^N (dy_i y_i^{\Lambda_1} (1 - y_i)^{\Lambda_2} (y_i - t)^n) (\Delta_N(y)^2)^\lambda \\ &= \frac{1}{S_n(V_1, V_2, 1/\lambda)} \int_{\mathcal{C}} \prod_{a=1}^n (dx_a x_a^{V_1} (1 - x_a)^{V_2} (1 - tx_a)^N) (\Delta_n(x)^2)^{1/\lambda} \end{aligned} \quad (\text{B1})$$

where the constants are related as

$$V_1 = \frac{\Lambda_1 + \Lambda_2 + 2}{\lambda} + N - 2 \quad V_2 = -\frac{\Lambda_2 + n}{\lambda} - N. \quad (\text{B2})$$

The normalization constant  $S_k(a, b, c)$  is given by the Selberg integral

$$S_k(a, b, c) = \prod_{j=0}^{k-1} \frac{\Gamma(a + 1 + cj)\Gamma(b + 1 + cj)\Gamma(1 + c(j + 1))}{\Gamma(a + b + 2 + c(k + j - 1))\Gamma(1 + c)}. \quad (\text{B3})$$

The integration contour  $\mathcal{C}$  encircles the cut between  $x_a = 0$  and  $x_a = 1$ . In our case (cf equation (19))

$$\Lambda_{1,2} = \lambda_{1,2} - \frac{1}{2} \quad V_1 = \frac{\lambda_1 + \lambda_2 + 1}{\lambda} + N - 2 \quad V_2 = -\frac{\lambda_2 + n - 1/2}{\lambda} - N. \quad (\text{B4})$$

We call identity (B1) fermionic replica ‘generalized colour–flavour transformation’. Indeed, for the RMEs values of the parameters ( $\lambda = 1/2, 1, 2$  and special values of  $\lambda_{1,2}$ ) equation (B1) essentially coincides with the fermionic replica version of Zirnbauer’s colour–flavour transformation [10, 23], after proper parametrization of the symmetric space elements and integration out of the irrelevant angles. The questions whether there is a geometrical ‘dual pair’ interpretation of equation (B1) for arbitrary parameters, and whether there is a supersymmetric (say for rational  $\lambda$ ) or bosonic replica analogue, are currently open.

In the large- $N$  limit we collect all the terms having the  $N$ th power into  $\exp[-N \sum_{a=1}^n \mathcal{S}(x_a, t)]$ , where

$$\mathcal{S}(x, t) = -\log(1 - tx) - \log x + \log(1 - x). \quad (\text{B5})$$

We then look for the stationary points of the ‘action’  $S(x)$  given by solutions of  $\partial_x S = 0$ . A simple algebra gives for the stationary points

$$x_{\pm} = 1 \pm i\sqrt{\frac{1-t}{t}}. \quad (\text{B6})$$

We then magnify the vicinity of  $t = 0$  by introducing  $\theta$  as  $t = \sin^2(\theta/(2N)) \simeq \theta^2/(2N)^2$  and changing the integration variable  $x_a$  to  $\phi_a$  as

$$x_a = 1 - i\sqrt{\frac{1-t}{t}} e^{i\phi_a}. \quad (\text{B7})$$

The two saddle points (B6) are at  $\phi_a = 0, \pi$ . Taking the limit  $N \rightarrow \infty$ , one obtains for the action  $NS(x_a, t) \rightarrow -i\theta \cos \phi_a$ . A straightforward algebra yields

$$(\Delta_n(x)^2)^{1/\lambda} \rightarrow \theta^{-\frac{n(n-1)}{\lambda}} \prod_{a=1}^n e^{i\frac{n-1}{\lambda}\phi_a} \left[ \prod_{a>b}^n \sin^2\left(\frac{\phi_a - \phi_b}{2}\right) \right]^{1/\lambda}. \quad (\text{B8})$$

The closed contour  $\mathcal{C}$  in the  $x_a$ -plane can be taken to be a circle, so that  $\phi_a \in [0, 2\pi]$ . As a result, one obtains equation (21).

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